Problem – 1 : Show that if every element of the group G except the identity element is of order 2, then G is abelian.

Solution: Let a, b $\in$ G such that $a \neq e, b \neq e$

Then $a^2 = e, b^2 = e$.

Also ab $\in$ G and so $(ab)^2 = e$

Now $(ab)^2 = e$

$\Rightarrow$ ab ab = e

$\Rightarrow$a (ab ab) = a e b

$\Rightarrow a^2$ ba b^2 = ab

$\Rightarrow$ e ba e = ab

$\Rightarrow$ ba = ab

Hence G is abelian

Problem – 2: If a, b be any two elements of a group G, then ab and ba have the same order.

Solution: ab = e(ab), where e is the identity of G

= $(b^{-1}b)$ (ab), since $b^{-1}b = e$.

Thus ab = b^{-1}(ba) b

Now o(ba) = o[b^{-1}(ba)b] = o(ab).

Hence, ab and ba have the same order.

Problem: If a, b be any two elements of a group G such that $a^5 = e$ and $aba^{-1} = b^2$, where e is the identity of G. Show that o(b) = 1 or o(b) = 31.

Solution: We have, $ab\,a^{-1} = b^2$ --------------------- (1)

$(ab\,a^{-1})^2 = (b^2)^2$

$\Longrightarrow (ab\,a^{-1})\,(ab\,a^{-1}) = b^4$

$\Longrightarrow ab(a^{-1}a)ba^{-1} = b^4$

$\Longrightarrow abe\,ba^{-1} = b^4$

$\Longrightarrow ab^2a^{-1} = b^4$ --(3)

Similarly, $ab^4a^{-1} = b^8$ -- (4)

$ab^8a^{-1} = b^{16}$ --(5)

$ab^{16}a^{-1} = b^{32}$ --(6)

Now, $b^{32} = ab^{16}a^{-1}$

$= a\,(ab^8a^{-1})\,a^{-1}$

$= a^2b^8a^{-2}$, by ----- (4)

$= a^2\,(ab^4a^{-1})a^{-2}$

$= a^3\,b^4\,a^{-3}$, by ----- (3)

$= a^3\,(ab^2a^{-1})\,a^{-3}$

$= a^4\,b^2\,a^{-4}$, by ----- (2)

$= a^4\,(ab\,a^{-1})\,a^{-4}$

$= a^5\,b\,a^{-5}$, by ----- (1)

$= e\,b\,e^{-5}$, since $a^5 = e$

$= b$

Thus $b^{32} = b \Longrightarrow b^{31} = b$

$\Longrightarrow$ o (b)|31. Since 31 is a prime number.

We have o (b) = 1 or, o(b) = 31.